Pricing Derivatives Analytically in a Heteroscedastic VAR Model with Jumps*

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Abstract

We derive closed-form solutions for option and swaption prices in an economic environment with time-varying second moments and stochastic jumps. Stochastic jumps are an important feature of risk models as the 2008 credit crisis has shown that extreme shocks may happen unexpectedly if the market panics. Time-varying second moments capture the persistence in volatilities and correlations, as for instance witnessed after the crash. The analytical derivative prices facilitate the incorporation of options and swaptions in the estimating procedure, thereby enhancing the proper modeling of higher moments. We also derive an analytical expression for a hybrid option that protects against a simultaneous decline of stock prices and interest rates. We show that the implied correlation between stock returns and interest rates, and thereby the price of these options, should be higher than the historical correlation, as jumps are more frequent under the risk-neutral measure than they are under the physical measure.

Keywords: Time-varying volatilities and correlations, jumps, options, swaptions, hybrid options

JEL codes: E43, G13

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1 Introduction

The 2008 credit crisis has seriously awakened the financial services industry. Most financial practitioners, as well as academics, were caught by surprise as the financial disorder after the default of Lehman Brothers was unprecedented. Risk models widely used by the financial industry regarded the start of such a crisis as highly unlikely. Moreover, these models generally considered volatilities and correlations to be constant, whereas during the crisis both volatilities and correlations between asset classes became much more extreme. Although time variation in correlation had been documented before (see for instance Campbell and Ammer (1993), Ang and Bekaert (2004), Baele, Bekaert, and Inghelbrecht (2010), and Campbell, Sunderam, and Viceira (2009)), the events in 2008 imply that the benefits of diversification are sometimes much smaller than anticipated by standard models. As a consequence, the usefulness of risk models came into question.

To remedy part of these flaws of traditional risk models, new models have been developed in which the proper modeling of higher moments of the data is given more priority. Options and swaptions can help in identifying these higher moments as these derivatives offer at maturity the right but not the obligation to engage in an equity respectively interest rate contract, at a pre-specified price. As this choice is particularly valuable if the future is uncertain, these products contain useful information on volatility. Moreover, by including options and swaptions with different strike prices, the market perception of the whole distribution of stock returns and interest rates can in principle be taken into account. In order to be able to use this market information for the calibration of risk models, closed-form solutions for the option and swaption prices are necessary. Given the model parameters, derivatives can also be priced by simulation although this is computationally intensive. For the calibration of a model however, simulation is not feasible as new simulations have to be done for every possible parameter set.

In this paper, we derive closed-form solutions for options and swaptions for the new risk model of APG; see Van den Goorbergh, Molenaar, Steenbeek, and Vlaar (2011). This model contains two new features to better capture financial risk characteristics. First, stochastic jumps are introduced. These jumps represent a sudden loss of confidence of the market leading to a significant drop in the stock market, lower risk-free interest rates, and a severe increase in credit spreads. Second, two dominant volatility factors are identified, representing monetary (inflationary) respectively real uncertainty (affecting risk aversion). The conditional covariance of the normally distributed shocks in the system depends linearly on the relevance of these two sources of uncertainty in the economy. As the correlation pattern differs between environments with either high monetary or real uncertainty, the model generates time-varying conditional correlations.

Another consequence of the credit crisis is that more parties seem to be interested in buying protection against extreme market movements. For pension funds, the two main risks are a decline of either the stock market (affecting their assets) or interest rates (increasing their liabilities). These risks can be hedged by buying both put options and receiver swaptions. This however might be overly expensive as the loss on equity might already be compensated by a profit on bonds, or the other way around. As an alternative, we derive a closed-form solution for a hybrid option for which the payout is based on the combined movements of stock prices and interest rates.

The rest of this paper is organized as follows. Section 2 describes the model. First, the dynamics
of the main driving variables is shown, including the variance and jump specifications. Second, the
dynamics under the risk-neutral measure is derived. Section 3 provides the closed-form solutions
for options and swaptions maturing next period, and shows the implications for implied volatilities.
Section 4 gives the analytical solution for the hybrid option. It is shown that the fair price of the
hybrid should be relatively high as the implied correlation between stocks and interest rates is higher
under the risk-neutral measure than it is under the physical measure. Section 5 gives a closed-form
approximation for the price of derivatives with a maturity of more than one period. The main results
are summarized in Section 6.

2 Heteroscedastic VAR Model with Jumps

The state variables are modeled in a quarterly Vector AutoRegressive (VAR) system with time-varying
second moments and subject to stochastic jumps:

\[ x_{t+1} = c_t + \Gamma x_t + J_{t+1} \nu + \sum \zeta_{t+1} \]

\[ J_{t+1} \sim B(p) \]

\[ \zeta_{t+1} \sim N(0, J_0) \]

The intercept \( c_t \) depends on a deterministic inflation target of the central bank (which has been 2%
since 1985). The six state variables are inflation in the eurozone (\( \pi_t \)), 3-month Euribor (\( y_{t}^{(1)} \)),
the quarterly log excess return on the stock market (\( x_{s,t} \)), the dividend yield (\( dy_t \)), the credit spread
between US Baa-bonds and Treasuries (\( cs_t \)), and an unobserved variable to better fit long term bonds
(\( u_t \)). These variables are either of direct importance for asset allocation or included because they are
known to help predict excess returns on stocks and bonds; see, e.g. Campbell and Shiller (1988), Fama
and French (1989), Campbell, Chan, and Viceira (2003), Campbell and Viceira (2005), Brandt and
\( J_{t+1} \) is Bernoulli distributed with probability \( p \). The impact of the jumps is measured by the vector
of mean jump sizes (\( \nu \)). The jumps represent sudden changes in sentiment in the market. Although
there might be early warning indicators that signalled the economic crises from the past, we assume
the probability of a jump to be constant, as the next economic crisis might have a completely new
unexpected cause. Besides, from a technical point of view, this assumption is necessary to facilitate
closed-form solutions for term structures and options. The \( \zeta_{t+1} \) shocks are normally distributed. The
time variation in their volatility is captured by the diagonal matrix \( S_t \).

\[
\text{Diagonal}(S_t) = \alpha + \beta x_t = \begin{bmatrix}
 s_{m,t} \\
 s_{r,t} \\
 1.01 - \omega_m + \omega_m s_{m,t} \\
 1.01 - \omega_r + \omega_r s_{r,t} \\
 1.01 - \omega_1 + \omega_1 (\omega_1 m s_{m,t} + (1 - \omega_1 m)s_{r,t}) \\
 1.01 - \omega_2 + \omega_2 (\omega_2 m s_{m,t} + (1 - \omega_2 m)s_{r,t})
\end{bmatrix}, \quad 0 \leq \omega_i \leq 1
\]
where
\[
\begin{align*}
  s_{m,t} &= \alpha_m + \pi_t + \beta_{my} y^{(1)}_t, \\
  s_{r,t} &= \alpha_r - \beta_{rx} x_t + \beta_{rd} d_t + cs_t,
\end{align*}
\] 
\(\beta_{my} \geq 0\)
\(\beta_{rx}, \beta_{rd} \geq 0\)

Two volatility factors are identified: a monetary factor and a risk-aversion factor. The monetary factor \((s_{m,t})\) measures the uncertainty in monetary policy. The factor depends on inflation (used for normalization) and the short-term interest rate. The risk-aversion factor \((s_{r,t})\) depends on excess returns on the stock market, the dividend yield, and the credit spread (used for normalization). The other elements on the diagonal of \(S_t\) are linear combinations of these two volatility factors, subject to two conditions. First, in order to prevent these volatilities from approaching zero, a small positive intercept is added. This requirement allows a much more flexible price of risk specification for these shocks in the essentially affine model; see Duffee (2002). Second, each element on the diagonal of \(S_t\) is distinct. This condition helps to identify \(\Sigma\) from the time-varying conditional covariances of the state variables \((\Sigma S_t \Sigma^T + \rho(1 - \rho)\nu \nu^T)\). Time variation in correlations is due to the changing importance of the two volatility factors: monetary shocks leading to a positive stock-bond correlation, and risk-aversion (or ‘flight-to-safety’) shocks leading to a negative stock-bond correlation.

The log pricing kernel for this model \((m_t)\) is chosen in such a way that closed-form solutions can be obtained for essentially affine models for the nominal and real term structure.

\[
-m_{t+1} = \frac{\hat{y}^{(1)}_t}{400} + \frac{1}{2} \lambda_t^T \lambda_t + \lambda_t^T \zeta_{t+1} + \ln \left(1 - \rho + \rho e^{-\phi}\right) + J_{t+1}\phi
\]  

where the 400 comes from the fact that we express yields in annual percentages, and where

\[
\lambda_t = S_t^{-1/2} (\lambda_0 + \Lambda_1 x_t)
\]  

Given this dynamics, the log bond prices \((\hat{y}_t^{(n)})\) are affine functions of the state variables; see Van den Goorbergh, Molenaar, Steenbeek, and Vlaar (2011) for details

\[
-p^{(n)}_t = A_n + B_n^T x_t
\]  

where

\[
\begin{align*}
  A_n &= A_{n-1} + (c - \Sigma \lambda_0)^T B_{n-1} - \frac{1}{2} \alpha^T (\Sigma^T B_{n-1})^\otimes 2 + \ln \left(1 - \rho + \rho e^{-\phi}\right) - \ln \left(1 - \rho + \rho e^{-\phi - B_{n-1}^T \nu}\right) \\
  B_n &= l_y + (\Gamma - \Sigma \lambda_1)^T B_{n-1} - \frac{1}{2} \beta^T (\Sigma^T B_{n-1})^\otimes 2
\end{align*}
\]

with \(A_0 = 0\) and \(B_0 = 0\), and \(l_y\) is a vector to select Euribor from the state variables. If we express annualized yields as \(y^{(n)}_t = a_n + b_n x_t\), it follows that \(a_n = 400 A_n / n\), and \(b_n = 400 B_n / n\). Similar expressions are obtained for the real term structure.
2.1 The Risk-Neutral $\mathcal{Q}$ measure

Given the pricing kernel as in equation (3), an asset can be priced as $Z_t = E_t[Z_{t+1}e^{m_{t+1}}]$ where $Z_{t+1}$ is a function of the state variables, hence also a function of the path of the state variables, and can be written as $Z_t(\zeta_{t+1}, J_{t+1})$. To arrive at the risk-free $\mathcal{Q}$ measure, we aim to find a different probability space of the stochastic shocks $(\zeta_t^Q, J_t^Q)$, in which the log price deflator $m_t^Q$ is simply the log price of a risk free one period bond $p_{t}^{(1)}$ ($= -y_{t}^{(1)}/400$). For simplicity, we derive the $\mathcal{Q}$ measure for the time period from $t = [0, 1]$ and drop the $t = 1$-subscript as we first study only one period. The results can be generalized to start from any time period and to multiple periods.

As the stochastic shocks $\zeta$ are normally distributed, and the jumps $J$ has a Bernoulli distribution of two states, where $J = 1$ with probability of $p$ and $J = 0$ with probability of $1 - p$, we have

\[
Z_0 = E_0[Z(\zeta, J)e^m] = E_0[Z(\zeta, J)e^{\phi_{t}}(1 - E_{t}[(1 - p)e^{-\phi_{t}}J_{t}])]
\]

\[
= E_0[Z(\zeta, J)e^{\phi_{t}}]E_0[1 - E_{t}[(1 - p)e^{-\phi_{t}}J_{t}]]
\]

\[
= \frac{e^{\phi_{t}}}{1 - p + pe^\phi} \int (1 - p)Z(\zeta, 0) + pZ(\zeta, 1)e^{-\phi}e^{-\frac{1}{2}\zeta^T \zeta}d\zeta
\]

For simplicity, we also drop the normalization factor of the normal distribution.

By changing the variable $\zeta^Q = \zeta + \lambda_0$, the equation above becomes

\[
Z_0 = E_0[Z(\zeta, J)e^m] = \frac{e^{\phi_{t}}}{1 - p + pe^\phi} \int [(1 - p)Z(\zeta^Q, 0) + pZ(\zeta^Q, 1)]e^{-\frac{1}{2}\zeta^Q^T \zeta^Q}d\zeta^Q
\]

where

\[
m^Q = p_{t}^{(1)}
\]

and

\[
p^Q = \frac{pe^\phi}{1 - p + pe^\phi}
\]

Hence we have found a new probability measure $\mathcal{Q}$ with sample space $(\zeta^Q, J^Q)$ to calculate $Z_0$, where the log pricing kernel $m^Q$ in (7) is the log risk free bond price. The jump process remains a Bernoulli distribution of two states, with the probability of jump $J^Q = 1$ occurring changed to $p^Q$ in (8). Substitute $\zeta = \zeta^Q - \lambda$, and $p^Q$ into the VAR equation (1), we arrive at the VAR equation in the risk-neutral $\mathcal{Q}$ measure:

\[
x_{t+1} = c_t^Q + \Gamma^Qx_t + J_{t+1}^Q\nu + \Sigma^Qz_{t+1}^Q
\]

where $c_t^Q = c_t - \Sigma A_1$, $\Gamma^Q = \Gamma - \Sigma A_1$, $J_{t+1}^Q$ is Bernoulli distributed with probability $p^Q$, and $\zeta_{t+1}^Q$ is standard normally distributed. As jumps have a devastating impact on returns, marginal utility is likely to be the highest in those states. Therefore, we expect the jumps to have a higher weight under the risk-neutral measure, which is the case if $p^Q > p$.  

5
3 Analytical Pricing of Options and Swaptions with Maturity of One Period

Given the risk-neutral Q measure just derived, we show that we can find closed-form solutions to price option and swaption with maturity of a single time step, i.e., one quarter. What is special about maturity of a single time step is that in such case the stochastic volatility of the VAR dynamics regresses to a deterministic one as it only depends on the initial state $x_0$.

3.1 Closed-form Solutions to Single Period Option Pricing

Applying risk-neutral valuation, European call and put options can be priced as

$$Call_{t}^{(K,n)} = P_t^{(n)}E^Q[\max(Z_{t+n} - K, 0)]$$

$$Put_{t}^{(K,n)} = P_t^{(n)}E^Q[\max(K - Z_{t+n}, 0)]$$

where $P_t^{(n)} = e^{-\rho_t^{(n)} n / 400}$ is the price of a $n$-period risk-free bond, $E^Q$ denotes expectation in the risk-neutral Q measure, $K$ strike price, and $n$ maturity. We show that the equations above can yield closed-form solution for options with maturity of a single time step, i.e., $n = 1$.

Let $Z_{t+1}$ denote stock price at time $t + 1$. Starting from the initial state at time $t$, and assuming initial stock price to be unity, the excess return for the stock at $t + 1$ is

$$\ln(Z_{t+1}) + p_t^{(1)} = l_{xs}^T(c_t^Q + \Gamma^Q x_t + J_{t+1}^Q \nu + \Sigma \sqrt{S} \zeta_{t+1})$$

where $l_{xs}$ is a vector to select the excess stock return from the state variables. For simplicity, we drop the superscript Q for the normally distributed shocks $\zeta$. We evaluate the options in the case of a jump occurring and in the case of no jumps. For a call with a maturity of one time step,

$$Call_{t}^{(K,1)} = P_t^{(1)}E_t^Q[\max(Z_{t+1} - K, 0)] = P_t^{(1)} \left( (1 - P_t^Q)E_t^{Q,0}[\max(Z_{t+1} - K, 0)] + P_t^Q E_t^{Q,1}[\max(Z_{t+1} - K, 0)] \right)$$

$$= (1 - P_t^Q)E_t^Q[\max(c_t^{Q} + \Gamma^Q x_t + \Sigma \sqrt{S} \zeta_{t+1} - K, 0)] + p_t^Q E_t^Q[\max(c_t^{Q} + \Gamma^Q x_t + \Sigma \sqrt{S} \zeta_{t+1} - K, 0)]$$

where $E_t^{Q,0}$ and $E_t^{Q,1}$ denote time $t$ expectation, conditional on $J_{t+1}^Q = 0$ and $J_{t+1}^Q = 1$ respectively.

As the shocks $\zeta_{t+1}$ are normally distributed, integration leads to\(^1\)

$$Call_t^{(K,1)} = \left( 1 - p_t^Q \right) \left( e^{l_{xs}^T(c_t^Q + \Gamma^Q x_t) + \sigma_{xs}^2 / 2}N(d_{xs} + \sigma_{xs}) - P_t^{(1)}KN(d_{xs}) \right) + P_t^Q \left( e^{l_{xs}^T(c_t^Q + \Gamma^Q x_t) + \sigma_{xs}^2 / 2}N(d_{xs} + \frac{l_{xs}^T \nu}{\sigma_{xs}} + \sigma_{xs}) - P_t^{(1)}KN(d_{xs} + \frac{l_{xs}^T \nu}{\sigma_{xs}}) \right)$$

where $\sigma_{xs} = \sqrt{l_{xs}^T \Sigma S \Sigma l_{xs}}$, $d_{xs} = -\ln(K) - \frac{l_{xs}^T(c_t^Q + \Gamma^Q x_t)}{\sigma_{xs}}$, and $N(d_{xs})$ denotes the value of the

\(^1\)If the jump probability is zero, the no-arbitrage conditions imply that $l_{xs}^T(c_t^Q + \Gamma^Q x_t) + \sigma_{xs}^2 / 2 = 0$, and (11) reduces to the Black Scholes formula.
cumulative normal distribution function at \(d_{xs}\). Similarly for one period puts, we have
\[
P_{ul_t^{(K,1)}} = (1 - \Phi(G)) \left( \frac{1}{\sigma_{xs}} \sum_{x=1}^{\infty} e^{-\frac{x^2}{2}} \Phi\left( -\frac{1}{\sigma_{xs}} \frac{t}{\sigma_{xs}} \right) \right)
\]

\[
P_{ul_t^{(K,1)}} = (1 - \Phi(G)) \left( \frac{1}{\sigma_{xs}} \sum_{x=1}^{\infty} e^{-\frac{x^2}{2}} \Phi\left( -\frac{1}{\sigma_{xs}} \frac{t}{\sigma_{xs}} \right) \right)
\]

3.2 Approximate closed-form Solutions to Single Period Swaption Pricing

A receiver swaption with strike interest rate \(X\), tenor \(N\), and maturity \(n\) can be priced as
\[
Receiver_t^{(X,N,n)} = P_t^{(n)} \left( \max \left( X \sum_{i=1}^{N} e^{-A_i t + B_i^T x_{i+n}} + e^{-A_N - B_N^T x_{i+n}} - 1, 0 \right) \right)
\]

(13)

For swaptions, an approximation is needed to derive an analytical solution, even for the single-period swaption. The sum of discounted cash flows of a swap is approximated by a single cash flow by a first order Taylor expansion of the log cash flows around the initial state:
\[
Rec_t^{(X,N,n)} = P_t^{(n)} \left( \max \left( R_t e^{-D_t^T x_{i+n}} - 1, 0 \right) \right)
\]

(14)

where
\[
D_t = \sum_{i=1}^{N} G_i B_i / \sum_{i=1}^{N} G_i
\]

\[
R_t = \sum_{i=1}^{N} G_i e^{D_t^T x_i}
\]

and \(G_i\) is defined as
\[
G_i = \begin{cases} 
X e^{-A_i t + B_i^T x_i}, & i = 1, 2, ..., N - 1 \\
(X + 1) e^{-A_N - B_N^T x_i}, & i = N
\end{cases}
\]

(15)

\(G_i\) is a function of the initial state \(x_t\), tenor of the underlying swap \(N\), and strike of the swaption \(X\). So are \(D_t\) and \(R_t\). This approximation can be interpreted as replacing a coupon baring bond by the most representative zero coupon bond, and the swaption is effectively approximated by an option on a zero-coupon bond.

A closed-form solution for (14) can be found for swaption maturing in the next time period (quarter). Similar to the approach in evaluating options, we expand (14) in the case of a jump occurring and of no jumps:
\[
Rec_t^{(X,N,1)} = P_t^{(1)} \left[ \max \left( R_t e^{-D_t^T x_{i+1}} - 1, 0 \right) \right] + P_t^{(1)} \left[ \max \left( R_t e^{-D_t^T x_{i+1}} - 1, 0 \right) \right]
\]

(15)

\[
Rec_t^{(X,N,1)} = P_t^{(1)} \left[ \max \left( R_t e^{-D_t^T x_{i+1}} - 1, 0 \right) \right] + P_t^{(1)} \left[ \max \left( R_t e^{-D_t^T x_{i+1}} - 1, 0 \right) \right]
\]

(15)
Integrating over the normally distributed $\zeta_{t+1}$, the one period receiver is found to be

\[
Rec^{(X,N,1)}_t = P_t^{(1)} \left( (1 - p^Q) \left( R_t e^{-\int_t^T (\zeta_t^2 + \Gamma Q_x) \, dt} \right) + \right.
\]

\[
\left. p^Q \left( R_t e^{-\int_t^T (\zeta_t^2 + \Gamma Q_x) \, dt} \right) \right)
\]

\[
(16)
\]

where $\sigma_y = \sqrt{D_t^T \Sigma \Sigma D_t}$ and $d_y = \ln(R_t) - D_t^T (c Q_t + \Gamma Q_x)$, depending on the initial state $x_t$. For payer swaptions, we have

\[
Pay^{(X,N,1)}_t = P_t^{(1)} \left( (1 - p^Q) \left( N(-d_y) - R_t e^{-\int_t^T (\zeta_t^2 + \Gamma Q_x) \, dt} \right) + \right.
\]

\[
\left. p^Q \left( N(-d_y + D_t^T \nu) - R_t e^{-\int_t^T (\zeta_t^2 + \Gamma Q_x + \nu) \, dt} \right) \right)
\]

\[
(17)
\]

In Figure 1 we plot receiver and payer prices in (16) and (17) for the estimated parameters, together those from simulation (13). The analytical and simulated prices are almost identical. The small pricing errors result from the approximation of a coupon baring bond by a zero-coupon bond. Consequently, for swaptions with shorter tenors the approximation error is even smaller.

Figure 1: Analytical and simulated price for one quarter swaptions with a tenor of 30 years.

3.3 Empirical Results

In order to better capture the higher moments of the dynamics of the state variables, options and swaptions were included in the estimation procedure of the VAR model. Figure 2 shows that the model explains reasonably well the time-variation in implied volatility on the equity and interest rate swap markets. The correlation between the model’s prediction and the realization is high for both markets. In other words, allowing for time-varying volatilities leads to better risk analysis. However, the extreme realizations during the credit crisis are not captured by the model, especially for the

\[\text{Since a coupon bond has higher convexity than a zero-coupon bond of the same duration, ignoring the positive convexity correction term results in underpricing for receivers and overpricing for payers in the analytical approximation.}\]
Figure 2: Implied volatility of at the money derivatives maturing next quarter

The credit crisis can be considered truly exceptional though. A conclusive judgment regarding the exclusiveness is impossible due to the lack of appropriate market data, but the model provides an indication. Over the period 1973 up until 2001, the predicted implied volatility on the swaptions market was always less than 15%.

Table 1 shows the fit of the model for prices of all the derivatives included. It should be emphasized that the model also fits both nominal and real interest rates with maturities up to 50 years. The same parameters determine the nominal and real yield of all maturities, and the prices of the derivatives. Consequently, these individual $R^2$ values can be improved upon if fewer variables are fitted by the model. If, for instance, nominal yields above 30 years and real yields are ignored, a better fit would be obtained for the derivatives. In the current setting, the model explains 32% (out of the money puts) to 63% (out of the money calls) of the price for equity options. With respect to swaptions, the fit declines somewhat with the maturity. For the at the money receivers (for which the strike equals the forward rate) 65% (10 year maturity) to 46% (30 year maturity) of the variance in the price is explained. For the 50 basis points out of the money receivers and payers, these percentages are somewhat lower.

A nice feature of the model is that it fits both derivative prices with high and low strikes. The model accounts for the volatility skew in options and swaptions, where implied volatilities decrease monotonically with increasing strikes; see Figure 3. It should be noted though that the reason for the

<table>
<thead>
<tr>
<th>Equity options</th>
<th>Receivers</th>
<th>Payers</th>
</tr>
</thead>
<tbody>
<tr>
<td>$Put^{(0.9,1)}$</td>
<td>0.32 (1994:III)</td>
<td>0.60 (2004:IV)</td>
</tr>
<tr>
<td>$Call^{(1,1)}$</td>
<td>0.49 (1994:III)</td>
<td>0.49 (2004:IV)</td>
</tr>
<tr>
<td>$Call^{(1,1,1)}$</td>
<td>0.63 (1994:III)</td>
<td>0.36 (2004:IV)</td>
</tr>
</tbody>
</table>

Notes: The first observation is shown in parenthesis.
skew differs between options and swaptions. For options the downward sloping skew is generated by the jumps, as in Andersen and Andreasen (2000). Conditional on a jump, the stock market drops by an additional 20% causing fat tails at negative returns; see Figure 4. The distribution of stock returns at one quarter is given, which indeed shows a strong skew. The reason this skew is so pronounced is that the relevant distribution for pricing options is under $Q$, not under $P$. As jumps have a strong negative impact on marginal utility, the probability of jump is much higher (about 13%) under the risk-neutral measure than under physical one (4%). For swaptions the jumps are less important, as interest rates only drop by about 25 basis points conditional on a jump. The skew follows directly from the Black (1976) model. In the Black model, forward swaps are assumed to have a log-normal distribution, while in the VAR model, zero-coupon interest rates and thus forward swaps are modeled as normally distributed with jumps. Given a close to normal distribution of forward swap rates, a downward sloping volatility skew always results when backed out from the Black model, even in the absence of jumps.

With respect to the put-call parity, our option model gives exactly the same volatility for options,
but not for swaptions. The deviation has to do with the small pricing errors due to the approximation of the coupon bearing bond by a zero-coupon bond. For swaptions that are far in the money the volatility can not be backed out easily as these derivatives are relative insensitive to the volatility.

4 Hybrid Options

An investor in stocks and bonds who wants to limit the downside risk on his portfolio can of course buy both put options and swaptions. This however can be overly expensive as the loss on equity might already be compensated by a profit on bonds, or the other way around. An attractive alternative might be to buy a hybrid option for which the payout is based on the combined movements of stock prices and interest rates. Whether the hybrid option is indeed attractive depends on the implied correlation that is used to price the hybrid. As the model in Section 2 is one of the first in which correlations are time-varying, it offers an interesting setting to determine a market-consistent price.

Rabobank pension fund was a pioneer to use these products on a large scale; see Van Capelleveen (2008). As a consequence, they were much less affected by the 2008 credit crisis. For a pension fund lower interest rates are generally bad news as their liabilities can be seen as a short position in a long term bond. In the case of the Rabobank pension fund, the risk of a joint decline in stock prices and interest rates was hedged by means of an equity linked swaption (ELS). This ELS is a receiver swaption for which the strike decreases with the stock return. Consequently, higher pension liabilities due lower interest rates only lead to a payout of the ELS if they are not already compensated by higher stock returns.

A technical challenge for pricing the ELS is that swaptions imply a stream of cash flows. In order to find an analytical solution, we had to approximate the sum of discounted cash flows by a single cash flow. For the ELS, this becomes impossible as the equity linked strike makes the weighting functions stochastic as well. In order to facilitate a closed-form solution, we will price a somewhat different hybrid option with similar characteristics, namely an interest linked put (ILP).

Replacing the dynamics of the stock return in (12) by the combined dynamics of the stocks and interest rates, whereby \( y^{(N)} = a_N + b_N^T x \), gives us the formula for the price of the single period ILP:

\[
\ln(K) = \ln(K_0) - \psi \left( y^{(N)} - y_0^{(N)} \right)
\]

(18)

Replacing the dynamics of the stock return in (12) by the combined dynamics of the stocks and interest rates, whereby \( y^{(N)} = a_N + b_N^T x \), gives us the formula for the price of the single period ILP:

\[
ILP_t^{(K_0, \psi, N, 1)} = (1 - p^Q) \left( P_t^{(1)} K_0 \mathcal{N}(-d_h) - e^{(l_{zs} + \psi b N)^T (c_Q^2 + \Gamma^2 z_t) - \psi b N^T z_t + \sigma^2 h / 2} \mathcal{N}(-d_h - \sigma h) \right) + p^Q \left( P_t^{(1)} K_0 \mathcal{N}(-d_h - \psi b N)^T \nu / \sigma_h \right)
\]

(19)

斥 counterpart, a zero-coupon equity linked swaption could be priced.
where $\sigma_h = \sqrt{(l_{x,x} + \psi b_N)^T \Sigma S_t \Sigma^T (l_{x,x} + \psi b_N)}$, and $d_h = -\frac{\ln(K_0) - p^{(1)}_t + (l_{x,x} + \psi b_N)^T (c_N^2 + \Gamma_N^2 x_t) - \psi b_N^T x_t}{\sigma_h}$.

Figure 5 compares the theoretical price over the past of the $ILP^{(0.9,0.2,60,1)}_t$ according to our model with the price of its constituents. The ILP we consider is 10% out of the money and has an interest rate sensitivity ($\psi$) of 0.2 to the 15 year interest rate. The constituents we compare the ILP with, give exactly the same payoff as the ILP at maturity if either stocks or interest rates do not move. For the stock market effect, this means the price of a normal put option. For the interest rate protection, we compute the artificial interest rate option:

$$AIO_t^{(K_0,\psi,N,1)} = p_t^{(1)} \left( (1 - p_t^2) \left( K_0 N(-d_i) - e^{\psi b_N^T (c_N^2 + \Gamma_N^2 x_t - x_t) + \sigma_t^2/2} N(-d_i - \sigma_i) \right) + p_t^Q \left( K_0 N(-d_i) - \psi b_N^T \left( K_0 N(-d_i) - \frac{\psi b_N^T \left( (c_N^2 + \Gamma_N^2 x_t + \nu) - x_t \right) + \sigma_t^2/2} N(-d_i - \frac{\psi b_N^T \nu}{\sigma_t} - \sigma_i) \right) \right) \right)$$

where $\sigma_i = \psi \sqrt{b_N^T \Sigma S_t \Sigma^T b_N}$, and $d_i = -\frac{\ln(K_0) + \psi b_N^T (c_N^2 + \Gamma_N^2 x_t)}{\sigma_i}$.

Figure 5 shows the ILP can either have a higher or lower price than its constituents. As long as interest rates and stock returns are expected to move into the same direction, the ILP provides more protection, and should therefore be more expensive. If on the other hand the correlation between the two is negative, buying separate derivatives gives a higher expected payoff, and the ILP is likely to be less expensive. In our model, negative correlation is primarily caused by monetary uncertainty. Consequently, ILP’s are relatively cheap in the high interest rate / inflation periods of the beginning of the seventies, eighties and nineties. During these episodes swaptions are relatively expensive compared to stock options.

Positive correlation can have two causes in our model. First real uncertainty induces flight to safety effects causing positive correlation between interest rates and stock returns. This time-varying risk aversion factor is especially prominent during crises. A second source for positive correlation are

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4We choose the 15 year rate ($N = 60$), as the average duration of the liabilities of Dutch pension funds is about 15 year. For the average pension fund the impact on its funding ratio of a 10% loss on the stock market is about the same as a 50 basis point decline in interest rates.
the jumps, since jumps lead to a simultaneous drop in stock prices (by 20%) and risk free interest rates (by 25 basis points). As the assumed jump probability and jump size are constant, this latter effect becomes more important if the volatility on the markets is low, for instance during the second half of the nineties and in the years before the credit crisis. According to our model, an ILP should have been relatively expensive in those years. The impact of the jumps on the implied correlation is relatively large as the jump probability under the risk-neutral measure (8) is about three times as large (13%) as under the physical measure (4%). Consequently, the implied correlation for the ILP is generally higher than the historical correlation between stocks and interest rates; see Figure 6. This might be one of the reasons why hybrid options are not more widely used: The implied correlation to calculate a fair price seems to be too high given historical observations. For an asset-only investor on the other hand, buying hybrid options (with $\psi$ negative) might be attractive, as the relative high implied correlation between stocks and interest rates, implies a relatively low price for ILP’s with a negative $\psi$.

The impact of the jumps is especially important for the tails of the distribution. For hybrids that are less far out of the money, the “normal” correlations are more important, and the ILP will be less expensive than the sum of its constituents. In practice however, protection is usually only bought for the extreme cases as some risk has to be taken in order to make higher expected returns.

5 Analytical Approximation to Price Derivatives of Longer Maturities

Most derivatives in portfolio will generally have a maturity of more than just one quarter. Due to the stochastic volatility, it is not possible to find exact closed-form solutions to the integrations in (10) for options and (14) for swaptions. Moreover, even for a simplified model with deterministic volatilities, the jumps effectively prohibit practical analytical results as the number of terms for an $n$-period derivative would be $2^{n+1}$. Schrager and Pelsser (2005) show that a closed-form approximation for swaptions can be found based on square-root dynamics for swap rates where the square-root dynamics has low variance and is replaced by its time zero expectation. We find that the stochastic
second moments in the heteroscedastic VAR with jumps also have lower variance, and that the impact of the jumps declines with the maturity. Therefore, we can approximate the stochastic volatility by its expectation conditional on the initial states, and replace the jump diffusion by a normal distribution. As such, an analytical approximation of the standard deviation for state variables $x_{t+n}$, and hence also for stock returns $\ln(Z_{t+n})$, can be found, and we can integrate analytically equations (10) and (14), with results similar to the Black and Scholes (1973) option model respectively the Black (1976) swaption model.

Assuming the variance of the second moments of the state variables in the heteroscedastic VAR model with jumps is small, as an approximation, $\Sigma_{t+n}\Sigma^{\top}$ is replaced by its expectation $\Sigma_{S_{t+n}^\top \Sigma^{\top}}$ conditional on $x_t$, where $S_{t+n} = \text{Diagonal}(\alpha + \beta x_t)$, and $x_{t+n} = c_{Qt} + \Gamma_{Q} x_{t+n-1} + J_{Qt+n}$.

Replacing the second moments with its expectation, the VAR equation is approximated as

$$x_{t+n} = c_{Qt} + \Gamma_{Q} x_{t} + J_{Qt+n} + \Sigma_{S_{t+n}^\top \Sigma^{\top}} \zeta_{t+1}$$  \hspace{1cm} (21)

Substituting $x_t$ recursively, we have

$$x_{t+n} = \sum_{i=0}^{n-1} \left( (\Gamma_{Q})^{i} \left( c_{Qt} + J_{Q_{t+n-i}} + \Sigma_{S_{t+n-i}^\top \Sigma^{\top}} \zeta_{t+n-i} \right) \right) + (\Gamma_{Q})^{n} x_{t}$$  \hspace{1cm} (20)

As the expected variance of $x_{t+1}$ conditional on time $t$ information is given by $\Sigma_{S_{t}^\top \Sigma^{\top}} + p^{Q} (1 - p^{Q}) \nu \nu^{\top}$, and since the $(\zeta_{t+i}, J_{Q_{t+i}})$ shocks are independently distributed, the $n$-period conditional variance of $x_t$ can be approximated by

$$\Omega_{t+n|t} = \sum_{i=0}^{n-1} \left( (\Gamma_{Q})^{i} \left( \Sigma_{S_{t+n-i}^\top \Sigma^{\top}} + p^{Q} (1 - p^{Q}) \nu \nu^{\top} \right) (\Gamma_{Q})^{i} \right)$$  \hspace{1cm} (22)

For the swaption prices with maturity $n$ only the state vector at $t + n$ matters. After approximating the stream of cash flows from the swap by a single cash flow, (20) and (22) serve as inputs for the Black formula to calculate the swaption price. For the options values, we need to sum the excess stock returns and risk-free returns between $t$ and $t+n$. This makes the variance slightly more complicated as we have to take the time-dependence in returns into account. The variance of the $n$-period return is given by

$$\sigma_{t+n|t}^{2} = \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{k=1}^{\min(i,j)} \left( H_{i} \left( (\Gamma_{Q})^{i-k} \left( \Sigma_{S_{t+k-1}^\top \Sigma^{\top}} + p^{Q} (1 - p^{Q}) \nu \nu^{\top} \right) (\Gamma_{Q})^{i-k} \right) H_{j}^{\top} \right)$$  \hspace{1cm} (23)

where

$$H_{i} = \begin{cases} l_{y} + l_{xs}, & 1 \leq i \leq n - 1 \\ l_{xs}, & i = n \end{cases}$$
The option price can subsequently be determined by the Black Scholes formula.

When comparing the approximated analytical prices with simulated ones, the difference for swaptions are generally very small. As in the case of the one-period swaptions, the small approximation error comes mainly from the approximation of a coupon bearing swap to a zero coupon bond. For options, the pricing errors are somewhat bigger due to the jumps, but still below 5%. These results confirm that the approximations are valid and the analytical approximation results in fairly accurate pricing of derivatives in the heteroscedastic VAR model with jumps.

6 Conclusion

In an economic environment with time-varying second moments and stochastic jumps modeled as a VAR system, we find closed-form solutions for options and swaptions with maturity of a single period. The time-varying second moments are due to the changing importance of the monetary and real uncertainty in the economy, and cause the correlation between stock and bond returns to vary over time. The stochastic jumps capture unexpected shocks when markets panic. The analytical pricing of options and swaptions enable the incorporation of derivatives in the estimation procedure, thereby enhancing the proper modeling of the risk characteristics of assets.

We also derive an analytical expression for a hybrid option, a put option with an exercise price linked to an interest rate (ILP). It can protect against a simultaneous decline of both stock prices and interest rates. This is of special interest to pension funds which is typically long stock and short bonds (liability) to protect their funding ratio. We show that the ILP can have a higher or lower price than its two components, depending on the co-movements of stocks and interest rates. ILP’s should be relatively cheap if monetary uncertainty is high as monetary shocks lead to negative correlation between stock returns and interest rates. Real uncertainty on the other hand leads to positive correlation and thereby a relatively high price for the ILP. Generally, the implied correlation between stock returns and interest rates, and thereby the price of these options, should be higher than the historical correlation, as jumps are more frequent under the risk-neutral measure than they are under the physical measure.

Most derivatives in a portfolio have maturities longer than one quarter since most asset managers, especially those of pension funds have longer investment horizon. VAR models are used to generate scenarios of asset returns, but pricing derivatives by simulation is computationally prohibitive. We show we can price options and swaptions of longer maturities with an analytical approximation with limited errors. Thereto, we replace the stochastic second moments of the VAR system by its expectation conditional on the initial state, and approximate the Bernoulli normal mixture by a normal distribution.
References


